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Huang, Q. (2002). On the Regularity of Solutions to Fully Nonlinear Elliptic Equations via the Liouville Property. *Proceedings of the American Mathematical Society*, 130 (7), 1955-1959.  
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# ON THE REGULARITY OF SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS VIA THE LIOUVILLE PROPERTY

QINGBO HUANG

(Communicated by David S. Tartakoff)

**ABSTRACT.** We show that any  $C^{1,1}$  solution to the uniformly elliptic equation  $F(D^2u) = 0$  must belong to  $C^{2,\alpha}$ , if the equation has the Liouville property.

## §1. INTRODUCTION

In this paper, we consider the interior regularity of solutions to the following fully nonlinear elliptic equation:

$$(1) \quad F(D^2u) = 0.$$

We assume that  $F$  is uniformly elliptic, i.e., there exist constants  $0 < \lambda \leq \Lambda$  such that

$$(2) \quad \lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|, \quad \text{for } M, N \in \mathcal{S}, \ N \geq 0,$$

where  $\mathcal{S}$  denotes the space of real  $n \times n$  symmetric matrices and  $\|N\|$  denotes the norm of  $N$ .

For simplicity, we also assume that  $F(0) = 0$ .

There have been a number of works concerning equation (1). For instance, see [CC], [GT], [K] and the references cited there. When  $F$  is a concave or convex functional, it is well known that the Evans-Krylov estimate

$$[D^2u]_{C^\alpha(B_{1/2})} \leq C \|u\|_{C^{1,1}(B_1)}$$

holds, and  $C^{1,1}$  viscosity solutions of (1) are  $C^{2,\alpha}$  for some  $\alpha > 0$ .

On the contrary, in the case when  $F$  is not concave nor convex,  $C^{1,1}$  viscosity solutions of (1) may not be in the  $C^2$  class. This has recently been shown by Nadirashvili in [N] in which he found a  $C^{1,1}$  viscosity solution  $u$  to the equation  $F(D^2u) = 0$  where  $F$  is smooth, uniformly elliptic and  $u$  is not  $C^2$ . Therefore, it would be interesting to know under what condition a  $C^{1,1}$  solution of (1) is actually in the  $C^2$  class.

It is our purpose in this paper to show that any  $C^{1,1}$  viscosity solution of (1) must be  $C^{2,\alpha}$  if the elliptic operator  $F$  has the Liouville property.

A continuous function  $u(x)$  is said to be a viscosity subsolution (*resp.*, supersolution) of (1) in a domain  $\Omega$  if for  $x_0 \in \Omega$  and  $\phi(x) \in C^2$ ,  $u - \phi$  attains the local

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Received by the editors September 20, 1999.

2000 *Mathematics Subject Classification.* Primary 35J60; Secondary 35B65.

*Key words and phrases.* Fully nonlinear elliptic equation, regularity, Liouville property, VMO.

maximum (*resp.*, minimum) at  $x_0$ , then  $F(D^2\phi(x_0)) \geq 0$  (*resp.*,  $\leq 0$ ). If  $u$  is both a subsolution and a supersolution, then we say  $u$  is a viscosity solution. We mention that if  $u \in C^{1,1}$ , then  $u$  is a viscosity solution of (1) if and only if  $u$  is a strong solution to (1).

Equation (1) or  $F$  is said to satisfy the Liouville property if  $u \in C_{loc}^{1,1}(\mathbf{R}^n)$  is an entire viscosity solution of (1) with bounded  $D^2u$  in  $\mathbf{R}^n$ ,  $|D^2u| \leq C$ , then  $u$  must be a polynomial of degree at most 2.

Let  $B_r(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < r\}$ .

Now we state the main theorem.

**Theorem.** Suppose that  $F \in C^1$  satisfies (2) and  $F(0) = 0$ . Let  $u \in C^{1,1}(B_1(0))$  be a viscosity solution of (1) in  $B_1(0)$ . If equation (1) satisfies the Liouville property, then for any  $0 < \alpha < 1$ ,  $u \in C^{2,\alpha}(B_{1/2}(0))$  and  $[D^2u]_{C^\alpha(B_{1/2}(0))} \leq C$ , where  $C$  depends only on  $n$ ,  $\lambda$ ,  $\Lambda$ ,  $\alpha$ ,  $\|u\|_{C^{1,1}(B_1(0))}$ ,  $F$ , and the modulus of continuity of  $DF$ .

## §2. THE PROOF OF THE THEOREM

We will use the blow-up technique to prove the Theorem. The tool needed to obtain a subsequence of blow-up solutions converging in  $W_{loc}^{2,2}(\mathbf{R}^n)$  is the  $W^{2,\delta}$  estimate for nondivergent uniform elliptic equations. For the convenience of our readers, let us give a little more preliminary information.

Recall that  $u \in \text{BMO}(\Omega)$  is in  $\text{VMO}(\Omega)$  if

$$\eta_u(R, \Omega) = \sup_{\substack{x_0 \in \Omega \\ 0 < r \leq R}} \int_{B_r(x_0) \cap \Omega} |u(x) - u_{x_0,r}| dx \longrightarrow 0, \quad \text{as } R \longrightarrow 0,$$

where  $\int_A f dx$  denotes the average of  $f$  over  $A$  and  $u_{x_0,r}$  the average of  $u$  over  $B_r(x_0) \cap \Omega$ . We will call  $\eta_u$  the VMO modulus of  $u$  in  $\Omega$ .

Now let us recall the class  $\mathcal{S}$  of solutions of uniformly elliptic equations. For more details, see [CC]. Let  $\mathcal{A}_{\lambda,\Lambda}$  denote all symmetric matrices whose eigenvalues belong to  $[\lambda, \Lambda]$ . Define Pucci extremal operators  $M^+(M)$  and  $M^-(M)$  by

$$M^+(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \text{trace}(AM),$$

$$M^-(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \text{trace}(AM),$$

for  $M \in \mathcal{S}$ . It is easy to check that  $M^+$  and  $M^-$  are uniformly elliptic operators. A continuous function  $u$  is in class  $\mathcal{S}$  if  $M^-(D^2u) \leq 0$  and  $M^+(D^2u) \geq 0$  in the viscosity sense.

The following result on precompact sets in  $L^p$  is a local variant of Theorem 3.44 in [A].

**Proposition 1.** Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  and  $\mathcal{A}$  a bounded subset of  $L^p(\Omega)$ ,  $1 \leq p < \infty$ . For any domain  $D \subset \subset \Omega$ , if

$$\sup_{u \in \mathcal{A}} \int_D |u(x+h) - u(x)|^p dx \rightarrow 0, \quad \text{as } |h| \rightarrow 0,$$

then  $\mathcal{A}$  is precompact in  $L^p(D)$ .

Now let us prove the following lemma.

**Lemma 1.** Assume that  $F$  satisfies (2) and  $F(0) = 0$ . Then the following two statements are equivalent:

- (i) If  $u \in C^{1,1}(B_1(0))$  is a viscosity solution of (1) in  $B_1(0)$  and  $|D^2u| \leq M$  in  $B_1(0)$ , then  $D^2u \in VMO(B_{1/2}(0))$  and  $\eta_{D^2u}(R) \leq \eta(R)$ , where  $\eta_{D^2u}(R)$  is the VMO modulus of  $D^2u$  in  $B_{1/2}(0)$ ,  $\lim_{R \rightarrow 0^+} \eta(R) = 0$ , and  $\eta$  depends only on  $n, \lambda, \Lambda, F$ , and  $M$ .
- (ii)  $F$  satisfies the Liouville property.

*Proof.* (i) implies (ii). Let  $u \in C^{1,1}_{loc}(\mathbf{R}^n)$  be an entire solution of (1) with  $|D^2u| \leq M$  in  $\mathbf{R}^n$ . Consider

$$v_k(y) = \frac{u(ky) - u(0) - Du(0)ky}{k^2}, \quad k = 1, 2, \dots$$

Obviously  $\|v_k\|_{C^{1,1}(B_1(0))} \leq C_n M$  and

$$F(D^2v_k) = 0 \quad \text{in } B_1(0).$$

Therefore by (i), for  $\rho > 0$  we have

$$\begin{aligned} & \int_{B_\rho(0)} |D^2u - (D^2u)_{0,\rho}| dx \\ &= \int_{B_{\frac{\rho}{k}}(0)} |D^2v_k - (D^2v_k)_{0,\frac{\rho}{k}}| dy \\ &\leq \eta_{D^2v_k}\left(\frac{\rho}{k}\right) \leq \eta\left(\frac{\rho}{k}\right) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned}$$

This implies that  $D^2u = \text{const}$  in  $\mathbf{R}^n$  and hence  $u$  is a polynomial of degree at most 2.

Suppose that  $F$  satisfies the Liouville property. We want to show (i). Let

$$X_M = \{u \in C^{1,1}(B_1(0)) : F(D^2u) = 0 \text{ and } |D^2u| \leq M \text{ in } B_1(0)\}.$$

To prove that (i) holds, it suffices to show the following claim:

$$(3) \quad \sup_{\substack{u \in X_M \\ x_0 \in B_{1/2}(0) \\ r \leq R}} \int_{B_r(x_0)} |D^2u - (D^2u)_{x_0,r}|^2 dx \longrightarrow 0, \quad \text{as } R \longrightarrow 0.$$

We will show (3) by contradiction. If (3) is false, then there exist  $\varepsilon_0 > 0$ ,  $r_k \longrightarrow 0$ ,  $x_k \in B_{1/2}(0)$ ,  $u_k \in X_M$  such that for  $k \geq 1$

$$\int_{B_{r_k}(x_k)} |D^2u_k - (D^2u_k)_{x_k,r_k}|^2 dx \geq \varepsilon_0.$$

Let

$$\begin{aligned} T_k y &= x_k + r_k y, & \Omega_k &= T_k^{-1} B_1(0); \\ v_k(y) &= \frac{u_k(x_k + r_k y) - u_k(x_k) - Du_k(x_k)r_k y}{r_k^2}. \end{aligned}$$

It is easy to check that

$$F(D^2v_k) = 0, \quad \text{in } \Omega_k.$$

$$(4) \quad \int_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2 dy \geq \varepsilon_0.$$

$$(5) \quad \|v_k\|_{C^{1,1}(B_{2A}(0))} \leq C_{n,A} M, \quad \text{if } B_{2Ar_k}(x_k) \subset B_1(0).$$

Now we want to show  $\{D^2v_k\}$  is precompact in  $L^2$ . By [CC] (see Prop. 5.5)

$$\Delta_{\tau e}v_k(y) = \frac{v_k(y + \tau e) - v_k(y)}{\tau} \in \mathcal{S}, \quad \text{in } B_{3A/2}(0), \quad |e| = 1.$$

By the  $W^{2,\delta}$  estimate ( $\delta > 0$ ) (see Prop. 7.4, [CC]) for functions in  $\mathcal{S}$

$$\int_{B_A(0)} |D^2\Delta_{\tau e}v_k|^\delta(y) dy \leq C_A \|\Delta_{\tau e}v_k\|_{L^\infty(B_{3A/2}(0))}^\delta \leq C_{A,M} M^\delta,$$

where  $\delta$  and  $C_A$  are independent of  $k$ .

Therefore, by (5) we have

$$\begin{aligned} & \int_{B_A(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^2 dy \\ & \leq C \|D^2v_k\|_{L^\infty(B_{2A}(0))}^{2-\delta} \int_{B_A(0)} |D^2v_k(y + \tau e) - D^2v_k(y)|^\delta dy \\ & \leq C\tau^\delta. \end{aligned}$$

By Proposition 1, this fact together with

$$\|D^2v_k\|_{L^2(B_{2A}(0))} \leq C_A$$

implies that  $\{D^2v_k\}$  is precompact in  $L^2(B_A(0))$ . Since  $v_k(0) = 0$ ,  $Dv_k(0) = 0$ , and  $\|v_k\|_{C^{1,1}(B_{2A}(0))} \leq C_A$ , we may assume that by the diagonalizing process

$$\begin{aligned} v_k &\longrightarrow v, & \text{in } W_{loc}^{2,2}(\mathbf{R}^n) \cap C_{loc}^1(\mathbf{R}^n), \\ D^2v_k &\longrightarrow D^2v, & \text{a.e. in } \mathbf{R}^n. \end{aligned}$$

Therefore,  $F(D^2v) = 0$  in  $\mathbf{R}^n$ . Since  $\|D^2v_k\|_{L^\infty(B_A(0))} \leq \|D^2u_k\|_{L^\infty(B_1(0))} \leq M$ ,  $|D^2v| \leq M$  in  $\mathbf{R}^n$ . By the Liouville property,  $v$  must be a polynomial of degree at most 2, and hence  $D^2v = \text{const.}$  This contradicts the following:

$$\int_{B_1(0)} |D^2v - (D^2v)_{0,1}|^2 dy = \lim_{k \rightarrow \infty} \int_{B_1(0)} |D^2v_k - (D^2v_k)_{0,1}|^2 dy \geq \varepsilon_0.$$

Thus, Lemma 1 follows.  $\square$

**Lemma 2.** *Let  $F \in C^1$  satisfy (2) and  $F(0) = 0$ . If  $u$  is a viscosity solution of (1) in  $B_1(0)$  and  $D^2u \in VMO(B_1(0))$ , then for any  $0 < \alpha < 1$ ,  $u \in C^{2,\alpha}(B_{1/2}(0))$  and  $[D^2u]_{C^{2,\alpha}(B_{1/2}(0))} \leq C$ , where  $C$  depends on  $n$ ,  $\alpha$ ,  $\lambda$ ,  $\Lambda$ , the modulus of continuity of  $DF$ ,  $\|u\|_{C^1(B_1(0))}$  and the VMO modulus of  $D^2u$ .*

*Proof.* By differentiating (1), we obtain

$$(6) \quad a_{ij}(x) D_{ij} \Delta_{he} u(x) = 0, \quad \text{in } B_{3/4}(0),$$

where  $\Delta_{he} u(x) = [u(x + he) - u(x)]/h$ ,  $h < \frac{1}{4}$ ,  $|e| = 1$ , and

$$a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1-\theta)D^2u(x) + \theta D^2u(x + he)) d\theta.$$

Let  $u_h(x) = u(x + he)$  and

$$c_{ij} = \int_0^1 \frac{\partial F}{\partial M_{ij}}((1-\theta)(D^2u)_{x_0,r} + \theta(D^2u_h)_{x_0,r}) d\theta.$$

Without loss of generality, we can assume that the continuity modulus of  $DF$  denoted by  $\omega(R)$  is concave. Obviously by Jensen's inequality

$$\begin{aligned} & \int_{B_r(x_0)} |a_{ij}(x) - c_{ij}| dx \\ & \leq \int_{B_r(x_0)} \int_0^1 \omega[|(1-\theta)(D^2u - (D^2u)_{x_0,r}) + \theta(D^2u_h - (D^2u_h)_{x_0,r})|] d\theta dx \\ & \leq \omega(\eta_{D^2u}(r)) \longrightarrow 0, \quad \text{as } r \longrightarrow 0. \end{aligned}$$

Therefore  $a_{ij} \in \text{VMO}(B_{3/4}(0))$ . Since (2) holds,  $\lambda'I \leq (a_{ij}) \leq \Lambda'I$  and (6) is uniformly elliptic. By the  $L^p$  estimate in [CFL], we obtain

$$\|\Delta_{he}u\|_{W^{2,p}(B_{1/2}(0))} \leq C\|\Delta_{he}u\|_{L^\infty(B_{3/4}(0))}.$$

Thus, we finish the proof of Lemma 2.  $\square$

The Theorem follows immediately from Lemma 1 and Lemma 2.

#### ACKNOWLEDGEMENT

The author expresses his gratitude to Professor L. Caffarelli for discussions on several occasions. The author also thanks the referee for some suggestions.

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